Indian Statistical Institute, Bangalore Centre
B.Math. (I Year) : 2014-2015
Semester II : Semestral Examination
Probability Theory II

08.05.2015 Time: 3 hours Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. State clearly the results you are using in your answers.

1. ($10 + 5 + 5 = 20$ marks) A machine has 2 components; lifespans of these two components are independent random variables, each having an exponential distribution with parameter 0.2. The machine works in such a way that as soon as the first component fails, the second component takes over and the machine continues to work. Let X denote the lifespan of the first component, and Y denote the lifespan of the machine.

   (i) Find the joint probability density function of (X, Y).

   (ii) What is the expected lifespan of the machine?

   (iii) Given that the first component failed at time 5, find the probability that the machine would still be working at time 15.

2. ($10 + 10 = 20$ marks) Let (X, Y) be absolutely continuous with probability density function

$$\begin{aligned} f(x,y) &= 3x, \quad \text{if} \quad 0 < y < x < 1, \\ &= 0, \quad \text{elsewhere.} \end{aligned}$$

   (i) Find the probability density function of $X - Y$.

   (ii) Find the conditional density of X given $Y = y$ for $0 < y < 1$.

3. ($14 + 6 = 20$ marks) Let (X, Y) have a bivariate normal distribution such that X and Y have standard normal distribution, and $0 \neq \rho \in (-1, 1)$ is the correlation coefficient between X and Y. Let $a \in \mathbb{R}$.

   (i) Find the distribution of $(X - aY, Y)$, indicating the marginal distributions and the correlation coefficient.

   (ii) Can $X - aY$ and Y be independent? Justify your answer.

4. (20 marks) Let $\varphi(\cdot)$ denote the characteristic function of a real valued random variable X. For any $a_j \in \mathbb{C},\ t_k \in \mathbb{R},\ 1 \leq j, k \leq n$ show that

$$\sum_{j,k=1}^{n} a_k \overline{a}_j \varphi(t_k - t_j) \quad \geq \quad 0.$$

(Here $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.)

5. (25 marks) For $a \geq 0, n = 1, 2, \cdots$ put

$$g(n, a) = e^{-n} \sum_{k \leq na} \frac{n^k}{k!}.$$

Using the central limit theorem, find $\lim_{n\to\infty} g(n, a)$. (*Hint:* Consider the cases $a < 1, a = 1, a > 1$ separately.)